MATH 2028 Conservative Vector Fields

GOAL: Study when does a line integral depend only on the endpoints of the curve but not on the particular path joining the endpoints.

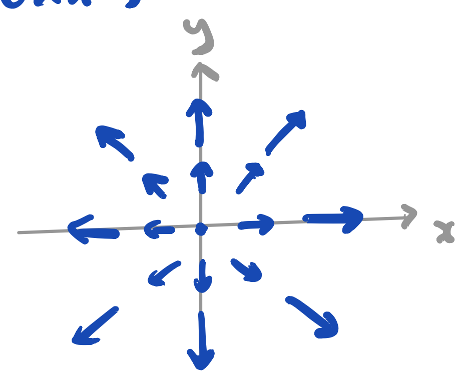
Recall: (Gradient vector field)

Given a C^1 function $f: \mathcal{U} \to \mathbb{R}$ defined on an open set $\mathcal{U} \subseteq \mathbb{R}^n$, we define its gradient $\nabla f : \mathcal{U} \to \mathbb{R}^n$ as the vector field on $\mathcal{U}$:

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$$

E.g.) $f(x,y) = x^2 + y^2$

$$\nabla f(x,y) = (2x, 2y)$$



Note: $\nabla(f + c) = \nabla f$ for any constant c.

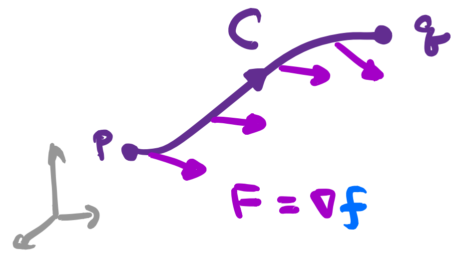
The following theorem is simply the Fundamental Theorem of Calculus for line integrals.

Prop: Let $F : U \to \mathbb{R}^n$ be a vector field s.t.

$$F = \nabla f \quad \text{on } U \subseteq \mathbb{R}^n$$

for some C^1 function $f : U \to \mathbb{R}$ defined on an open set $U \subseteq \mathbb{R}^n$. **THEN:** for **ANY** curve $C \subseteq U$ joining p to q, we have

$$\boxed{\int_C F \cdot d\vec{r} = f(q) - f(p)}$$



Remark: The R.H.S. for the above equality depends only on the value of the function f at the endpoints p, q but NOT the specific path C joining p to q. In other words, the line integral $\int_C F \cdot d\vec{r}$ is path-independent when F is a gradient vector field in $U \subseteq \mathbb{R}^n$.

Proof: The proof is simply the usual Fund. Thm. of Calculus on $\mathbb{R}$.

Let C be parametrized by $\gamma(t) : [a,b] \to \mathbb{R}^n$
s.t. $\gamma(a) = p$ and $\gamma(b) = q$.

$$\int_C F \cdot d\vec{r} = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$\overset{(F = \nabla f)}{=} \int_a^b \nabla f(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$\overset{(\text{Chain Rule})}{=} \int_a^b \frac{d}{dt}\Big(f(\gamma(t)) \Big)\, dt$$

$$\overset{(\text{Fund. Thm. of Calculus})}{=} f(\gamma(b)) - f(\gamma(a))$$

$$= f(q) - f(p)$$

$\blacksquare$

The following theorem gives a useful characterization of gradient vector fields.

<u>Thm</u>: Let $F : \mathcal{U} \to \mathbb{R}^n$ be a cts vector field on an open set $\mathcal{U} \subseteq \mathbb{R}^n$. <u>THEN</u>: the following are equivalent:

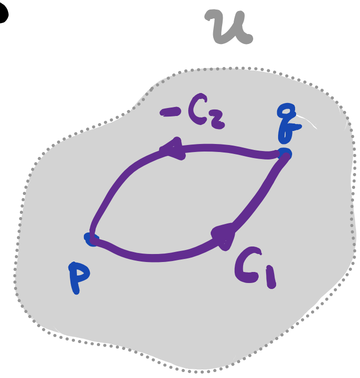
(1) $\oint_C F \cdot d\vec{r} = 0$ $\forall$ <u>closed</u> curve $C \subseteq \mathcal{U}$.

(2) $\int_C F \cdot d\vec{r}$ is <u>path-independent</u> in $\mathcal{U}$

(3) $F = \nabla f$ for some C^1 function $f : \mathcal{U} \to \mathbb{R}$.

Proof: $\underline{(1) \Rightarrow (2)}$: Suppose C_1, C_2 are two curves in $\mathcal{U}$ joining p to q. THEN, $C_1 \cup (-C_2)$ is a closed curve in $\mathcal{U}$. By (1),

$$0 = \oint_{C_1 \cup -C_2} F \cdot d\vec{r} = \int_{C_1} F \cdot d\vec{r} - \int_{C_2} F \cdot d\vec{r}$$

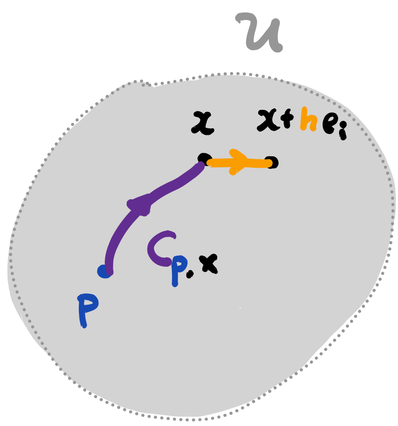


Hence, $\displaystyle\int_{C_1} F \cdot d\vec{r} = \int_{C_2} F \cdot d\vec{r}$

$\underline{(2) \Rightarrow (3)}$: WLOG, assume $\mathcal{U}$ is connected.

Fix $p \in \mathcal{U}$. We define a function $f : \mathcal{U} \to \mathbb{R}$ by

$$f(x) := \int_{C_{p,x}} F \cdot d\vec{r}$$



where $C_{p,x}$ is $\underline{\text{ANY}}$ curve in $\mathcal{U}$ connecting p to x.

(2) $\Rightarrow$ f is well-defined.

$\underline{\text{Claim}}$: $\nabla f = F$ in $\mathcal{U}$

i.e. $\dfrac{\partial f}{\partial x_i} = F_i$ where $F = (F_1, F_2, \dots, F_n)$

Let $C_{x,x+he_i}$ be the straight line segment joining x to $x+he_i$ for small values of h.

Then, $C_{p,x} \cup C_{x,x+he_i}$ is a piecewise C^1 curve joining p to $x+he_i$. Hence,

$$\frac{\partial f}{\partial x_i}(x) := \lim_{h\to 0} \frac{1}{h}\Big(f(x+he_i) - f(x) \Big)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\int_{C_{p,x}\cup C_{x,x+he_i}} F\cdot d\vec{r} - \int_{C_{p,x}} F\cdot d\vec{r} \right)$$

$$= \lim_{h\to 0} \frac{1}{h} \int_{C_{x,x+he_i}} F\cdot d\vec{r}$$

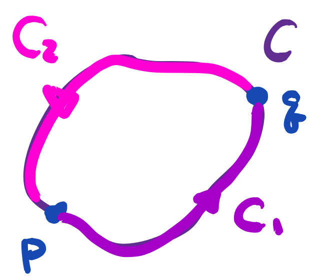
$$= \lim_{h\to 0} \frac{1}{h} \int_0^h \underbrace{F(x+te_i)\cdot e_i}_{F_i(x+te_i)}\, dt$$

$$= F_i(x)$$

<u>(3) ⇒ (1)</u>: Take any two points p, q on a closed curve C, then $C = C_1 \cup C_2$. THEN,

$$\oint_C F\cdot d\vec{r} = \int_{C_1} F\cdot d\vec{r} - \int_{-C_2} F\cdot d\vec{r} = 0$$

by previous Proposition.



$\blacksquare$

Def$^{\underline{n}}$: A vector field $F : \mathcal{U} \to \mathbb{R}^n$ is said to be conservative if $\oint_C F \cdot d\vec{r} = 0$ for ANY closed curve $C \subseteq \mathcal{U}$.

By Theorem above, F is conservative iff $\exists$ C^1 function $f : \mathcal{U} \to \mathbb{R}$, determined by F up to an additive constant if $\mathcal{U} \subseteq \mathbb{R}^n$ is connected, s.t.

$$F = \nabla f$$

We call f a potential function for F.

Example: Find a potential function f for the conservative vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

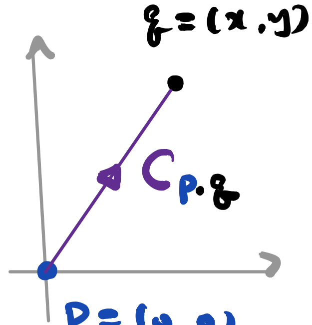
$$F(x, y) = (e^x + 2xy,\ x^2 + \cos y)$$

Solution 1: Fix $p = (0, 0)$. From the proof of Theorem above, we know that

$$f(q) = \int_{C_{p,q}} F \cdot d\vec{r}$$

where $C_{p,q}$ is ANY curve in $\mathcal{U} = \mathbb{R}^2$ joining $p = (0,0)$ to q.

In particular, if we take $C_{p,q}$ be the line segment from $p = (0,0)$ to $q = (x,y)$ parametrized by $\gamma(t) = (tx, ty)$, $0 \le t \le 1$.



$$\gamma'(t) = (x, y)$$

$$F(\gamma(t)) = (e^{tx} + 2t^2xy,\ t^2x^2 + \cos ty)$$

Hence,

$$f(x,y) = \int_0^1 x(e^{tx} + 2t^2xy) + y(t^2x^2 + \cos ty)\, dt$$

$$= \left[e^{tx} + x^2y\,t^3 + \sin ty \right]_{t=0}^{t=1}$$

$$= e^x + x^2y + \sin y - 1$$

It is easy to check that $\nabla f = F$:

$$\begin{cases} \dfrac{\partial f}{\partial x} = e^x + 2xy \\[2ex] \dfrac{\partial f}{\partial y} = x^2 + \cos y \end{cases}$$

i.e. $\nabla f = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right)$

$= F$

$\square$

Solution 2 : We can also find f by formally "integrating" $\mathbf{F}$. Recall that $\nabla f = \mathbf{F}$ means

$$\begin{cases} \dfrac{\partial f}{\partial x} = e^x + 2xy \\ \dfrac{\partial f}{\partial y} = x^2 + \cos y \end{cases}$$

Integrating the 1^{st} equation w.r.t. x, we have

$$f(x,y) = \int (e^x + 2xy)\,dx$$

$$= e^x + x^2 y + h(y)$$

for any arbitrary function h of y.

Plug into 2^{nd} equation,

$$x^2 + h'(y) = \frac{\partial f}{\partial y} = x^2 + \cos y$$

$\Rightarrow$ $h'(y) = \cos y$, i.e. $h(y) = \sin y + c$

for any constant c

Therefore, $f(x,y) = e^x + x^2 y + \sin y + c$

for any constant c. $\square$